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Approval voting: three examples

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Abstract In this paper we discuss three examples of approval voting games. The first one illustrates that a stronger solution concept than perfection is needed for a strategic analysis of this type of games. The second example shows that sophisticated voting can imply that the Condorcet winner gets no vote. The third example shows the possibility of insincere voting being a stable equilibrium.

Keywords Approval voting · Sophisticated voting · Sincere voting · Perfect equilibrium · Stable set

JEL Classification C 72 · D72

1 Introduction

Approval voting (AV) — a voting procedure in which voters may vote for as many candidates as they wish — has become an extremely popular voting

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system, being used in a large number of electoral contests.[1] Its proponents (for instance Brams 1980; Brams and Fishburn 1978, 1981, 2005) have discussed several advantages that it has over other electoral systems, and have even suggested that it is "the electoral reform of the twentieth century".

One advantage that AV is supposed to have over other voting systems is that it helps select the "strongest" candidate. Of course, the notion of what is the strongest candidate is not always well defined. But, if a candidate beats all other candidates in pairwise contests — that is, if it is a Condorcet winner — then it is intuitive to label this as the strongest candidate. It is known that plurality rule and several other systems will sometimes fail to elect the Condorcet winner. However, Fishburn and Brams (1981) prove that if a Condorcet winner exists then the AV game has a Nash equilibrium in undominated strategies that selects the Condorcet winner. A main purpose of this paper is to examine the "tendency" of AV to select Condorcet winners when they exist.

In order to do so, we use the model of one stage voting procedures developed by Myerson and Weber (1993) to analyse various features of AV games. Since Nash equilibrium has no predictive power in voting games such as the ones induced by AV when there are three or more voters, we focus on refinements of Nash equilibrium. In particular, a first example demonstrates that the perfect equilibrium solution concept is not restrictive enough in the context of approval games, since some outcomes induced by this concept are excluded by iterated elimination of dominated strategies and by strategic stability.[2] The terms *sophisticated voting* was introduced in the voting literature by Farquharson (1969). It refers to iterating the "exhaustive" elimination of all weakly dominated strategies by all players. In this example sophisticated voting results in an outcome in which the Condorcet loser and Condorcet winner are selected with the same probability. A second example shows how the Condorcet winner can get *no* vote at all according to sophisticated voting and strategic stability. This shows that the Fishburn and Brams (1981) result cannot be extended to these more demanding concepts.

A ballot or strategy under AV is said to be *sincere* for a voter if it shows no "hole" with respect to the voter's preference ranking: if the voter sincerely approves of a candidate x she also approves of any candidate she prefers to x. Much of the preceding work on AV has assumed that voters will only use sincere strategies. For instance, Niemi (1984) asserts that "under approval voting, voters are never urged to vote insincerely". It is true that for every pure strategy of the other players, the set of best replies contains a sincere strategy. However, this is no longer true when one considers mixed strategies. We construct another example of a strategy combination which is a strategically stable equilibrium and where an insincere strategy is the unique best-response for a voter. This example has another surprising property: the outcome corresponding to this strategy combination turns out to select the Condorcet loser with

[1] See Brams and Fishburn (2005) for an account of the various contexts in which AV is used.

[2] The same drawback of the perfect equilibrium concept holds with plurality rule. See De Sinopoli (2000) for an analysis of equilibrium refinements with such a voting rule.

the highest probability and the Condorcet winner with the lowest probability. Under plurality rule, for the same preferences, there is only one stable set and, in this stable set, the Condorcet winner is elected with probability 1.

Our paper does not contain any general results. However, the examples do suggest that it is important to subject the received wisdom about AV to closer scrutiny.

2 The framework

Let $C = \{1, ..., K\}$ be the set of candidates, and $N = \{1, ..., n\}$ be the set of voters. Under Approval Voting (AV), a ballot is a subset of the set of candidates. The approval voting rule selects the candidate receiving the maximum number of votes or "approvals". In case two or more candidates get the maximum number of votes, ties are broken by an equi-probable lottery on the set of tied candidates. Hence, every voter has 2^K pure strategies, corresponding to the set of vectors with K components, where each entry is either zero or one.[3]

The strategy space of each player is

$$\Sigma = \Delta (V)$$

where $V = \{0, 1\}^K$ is the set of pure strategies.

In order to determine the winner, we do not need to know the ballots cast by each voter - it is enough to know their sum. Given a pure strategy vector $v \in V^n$, let $\omega = \sum_{i=1}^{n} v^i$. Clearly ω is a K-dimensional vector, and each coordinate represents the total number of votes obtained by the corresponding candidate. Then, denoting by $p (c \mid v)$ the probability that candidate c is elected corresponding to v, we have

$$p (c \mid v) = \begin{cases} 0 & \text{if } \exists m \in C \text{ } s.t \text{ } \omega_c < \omega_m \\ \frac{1}{q} & \text{if } \omega_c \geq \omega_m \text{ } \forall m \in C \text{ and } \# \{d \in C \text{ } s.t. \text{ } \omega_c = \omega_d\} = q. \end{cases} \tag{1}$$

Each voter $i \in N$ has a VNM utility function characterized by $u^i : C \to \Re$, with u^i_c representing the payoff that player i gets if candidate c is elected. Hence, given the utility vectors $\{u^i\}_{i \in N}$, we have a normal form game. For each pure strategy combination v, the payoff of player i is given by

$$U^i (v) = \sum_{c \in C} p (c \mid v) u^i_c. \tag{2}$$

3 A "one" in the kth component denotes voting for candidate k.

Clearly, we can extend (1) and (2) to mixed strategies. Under a mixed strategy σ we have

$$p(c \mid \sigma) = \sum_{v \in V} \sigma(v)\, p(c \mid v)$$

and

$$U^i(\sigma) = \sum_{c \in C} p(c \mid \sigma)\, u_c^i,$$

where, as usual, $\sigma(v)$ denotes the probability of the (pure) strategy combination v under σ.

Since the election rule depends only upon the sum of the votes cast, the payoff functions and the best reply correspondences also have this property. Hence, the analysis will often refer to the following set:

$$\Omega^{-i} = \left\{ \omega_{-i} \mid \exists v \in V^n \text{ s.t. } \sum_{j \neq i} v^j = \omega_{-i} \right\}.$$

It is easy to see (cf. Brams and Fishburn 1978) that an *undominated* strategy always approves the most preferred candidate(s) and does not approve the least preferred one(s).

3 Example 1

We show here that in the example below, perfection is not an appropriate concept since the set of perfect equilibria includes strategy n-tuples (and induced outcomes) which do not survive iterated elimination of dominated strategies.

Example 1 There are six voters and three candidates. Utilities are given by

$$u^1 = u^2 = (3, 1, 0), u^3 = u^4 = (0, 3, 1), u^5 = u^6 = (0, 1, 3).$$

We first define the concept of perfect equilibrium.

Definition 1 *A completely mixed strategy σ^ε is an ε - perfect equilibrium if*

$$\forall i \in N,\ \forall v^i, \bar{v}^i \in V^i, \quad \text{if } U^i\left(v^i, \sigma^\varepsilon\right) > U^i\left(\bar{v}^i, \sigma^\varepsilon\right), \text{ then } \sigma^\varepsilon\left(\bar{v}^i\right) \leq \varepsilon.$$

A strategy combination σ is a perfect equilibrium if there exists a sequence $\{\sigma^\varepsilon\}$ of ε - perfect equilibria converging (for $\varepsilon \to 0$) to σ.

It is easy to see that the strategy combination

$$c = \Big((1, 0, 0), (1, 0, 0), (0, 1, 1), (0, 1, 1), (0, 0, 1), (0, 0, 1)\Big)$$

is an undominated equilibrium, leading to the election of the third candidate. We now show that c is a perfect equilibrium.

Proposition 2 *In the AV game for example 1, c is a perfect equilibrium.*

Proof Consider the following completely mixed strategy combination σ^{ε}, where ξ_i denotes the mixed strategy of player i which assigns equal probability to all his pure strategies.

$\sigma_i^{\varepsilon} = (1-8\varepsilon^2)(1,0,0) + 8\varepsilon^2\,(\xi_i) \quad i = 1,2$
$\sigma_i^{\varepsilon} = (1-8\varepsilon^2)(0,1,1) + 8\varepsilon^2\,(\xi_i) \quad i = 3,4$
$\sigma_i^{\varepsilon} = (1-\varepsilon-7\varepsilon^2)(0,0,1) + (\varepsilon-\varepsilon^2)\,(1,0,0) + 8\varepsilon^2\,(\xi_i) \quad i = 5,6$

It is easy to see that, for ε sufficiently close to zero, this is an ε -perfect equilibrium. Suppose all voters other than i choose the strategies prescribed by c. Then, the two undominated strategies of voter i are equivalent. Since for ε going to zero, the probability of player 5 (or 6) to tremble towards $(1,0,0)$ is infinitely greater than the probability of any other "mistake", due to the trembling of one or several players, it is enough to check that in this event the limiting strategy is preferred to the other undominated strategy.

Hence, for player 1, the relevant contingency which allows him to discriminate between his two undominated strategies is when the behavior of the others is summarized by the vector $\omega_{-1} = (2,2,3)$. Since

$$U^1((1,0,0) \mid (2,2,3)) = \frac{3}{2} > \frac{4}{3} = U^1((1,1,0) \mid (2,2,3))$$

approving only the most preferred candidate is the best reply to σ^{ε} for player 1. The same statement obviously applies for player 2.

For player 3, the relevant contingency in order to discriminate between his two undominated strategies is given by $\omega_{-3} = (3,1,2)$. Since

$$U^3((0,1,1) \mid (3,1,2)) = \frac{1}{2} > 0 = U^3((0,1,0) \mid (3,1,2)),$$

$(0,1,1)$ is the best reply to σ^{ε}. The same statement is true for player 4.

For player 5, the relevant event is given by $\omega_{-5} = (3,2,2)$ with

$$U^5((0,0,1) \mid (3,2,2)) = \frac{3}{2} > \frac{4}{3} = U^5((0,1,1) \mid (3,2,2)).$$

Hence $(0,0,1)$ is the best reply to σ^{ε}, and the same holds for player 6.

Therefore, $\{\sigma^{\varepsilon}\}$ is a sequence of ε *-perfect* equilibria. Since c is the limit of σ^{ε}, it is perfect. □

We now study the strategy combination

$$e = \big((1,0,0),(1,0,0),(0,1,0),(0,1,0),(0,0,1),(0,0,1)\big)$$

in which each voter approves only his most prefered candidate and that results in a complete tie between the three candidates.

Proposition 3 *In the AV game for example 1, let e be the strategy profile in which each voter approves only his most prefered candidate; e is the unique sophisticated equilibrium, moreover* $\{e\}$ *is the unique stable set of this game, hence survives any sequence of elimination of weakly dominated strategies.*

Proof Each voter has only two undominated strategies — approving only his most preferred candidate or approving the first two candidates in his preference ranking. Once all the dominated strategies have been eliminated, we have a reduced game with the following pure strategy sets:

$$\begin{aligned} V'^i &= \{(1,0,0),(1,1,0)\} \qquad i=1,2 \\ V'^i &= \{(0,1,0),(0,1,1)\} \qquad i=3,4 \\ V'^i &= \{(0,0,1),(0,1,1)\} \qquad i=5,6 \end{aligned}$$

In this reduced game, the last four voters have a unique dominant strategy — to approve only the most preferred candidate. For instance, consider player 3. In each ω_{-3} the first candidate gets two votes while the second gets at least one and the third at least two. Hence, except for $\omega_{-3} = (2,1,2)$, the approval of only the second candidate is either equivalent to the other strategy, since both lead to the election of the same candidate, or it is preferred. Moreover, if $\omega_{-3} = (2,1,2)$, the strategy $(0,1,0)$ results in all the 3 candidates being elected with equal probability. This yields an expected utility of $\frac{4}{3}$. If strategy $(0,1,1)$ is played, then candidate 3 is elected with probability one. Since this gives voter 3 a utility of 1, $(0,1,0)$ dominates $(0,1,1)$.

The same argument applies to the fourth voter and a symmetric one to the last two voters. Hence, we can further reduce the game by eliminating the strategy $v^i = (0,1,1)$ for $i = 3,4,5,6$. In this game, player 1 (resp. 2) can face only two circumstances, namely $\omega_{-1} = (1,2,2)$ or $\omega'_{-1} = (1,3,2)$. In the latter case, his two strategies are equivalent since both lead to the election of the second candidate; in the former case, $(1,0,0)$ is preferred to $(1,1,0)$, giving a utility of $\frac{4}{3}$ instead of 1. Hence $(1,0,0)$ is dominant for player 1 (resp. 2). Thus, iterated elimination of dominated strategy isolates the equilibrium e where each voter approves only his most preferred candidate.

Notice that e is *strict*, and hence, *isolated*. This implies that $\{e\}$ is the unique Mertens-stable set of the game.[4] This in turn implies that e survives any sequence of elimination of dominated strategies. □

The above results, namely that c is a perfect equilibrium but only $\{e\}$ is a stable set, holds for every game with the same preference order and such that, for every voter, the difference in utility between the most preferred candidate and the second preferred one is greater than the difference between the second and the least preferred one.

[4] See Mertens (1989) for a definition of this concept. We just recall that stable sets, which are connected set of perfect equilibria, always exist and that every stable set contains a stable set of every game obtained by iterated elimination of dominated strategies. These properties directly imply the claim.

Furthermore, the unique strategy combination surviving iterated elimination of dominated strategies elects all the three candidates with the same probability. In other words, the Condorcet loser (candidate 1) is elected with the same probability as the Condorcet winner (candidate 2)!

4 Example 2

In this section, we propose a more striking example in which sophisticated voting implies that nobody approves the Condorcet winner.

Example 2 There are three voters and four candidates. Utilities are given by

$$u^1 = (10, 0, 1, 3), \quad u^2 = (0, 10, 1, 3), \quad u^3 = (1, 0, 10, 3).$$

Note that at this profile, candidate 4 is the unique Condorcet winner.

Proposition 4 *In the AV game for example 2, let e be the strategy profile in which each voter approves only his most prefered candidate; e is the unique sophisticated equilibrium, moreover $\{e\}$ is the unique stable set of this game, hence survives any sequence of elimination of weakly dominated strategies. At e the Condorcet winner receives no vote.*

Proof The fourth alternatives defeats any other by a strict majority (2 votes against 1), hence is the unique Condorcet winner.

Recall that an undominated strategy always approves the most preferred candidate and does not approve the least preferred one. Hence, every voter has only four undominated strategies. After we eliminate all the others, it is easy to verify that, for player 1, the strategy $(1, 0, 0, 0)$ dominates $(1, 0, 1, 1)$ and $(1, 0, 1, 0)$. Once these two strategies of player 1 are eliminated, $(0, 0, 1, 0)$ is dominant for player 3. In the reduced game, player 3 has only one strategy - $(0, 0, 1, 0)$, while player 1 has two, namely $(1, 0, 0, 0)$ and $(1, 0, 0, 1)$. Now, $(0, 1, 0, 0)$ is dominant for player 2. Hence eliminating the other strategies, $(1, 0, 0, 0)$ becomes dominant for player 1. It is easy to see that this equilibrium is strict; the result follows like in the previous proposition. □

Hence, sophisticated voting (and thus strategic stability) may imply that the Condorcet winner receives no approval vote. As we have remarked earlier, Fishburn and Brams (1981) prove that if a candidate x is a Condorcet winner, then there is a sincere undominated strategy combination that elects x. This and the previous example show how this result cannot be extended to sophisticated (or strategically stable) strategies.[5]

[5] In this example the undominated equilibrium electing the Condorcet winner is $((1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1))$. It can be proved that such an equilibrium is not even perfect. Hence the exclusion of the "Condorcet outcome" from the solution set does not depend on the definition of stability. As a matter of fact, not even a weaker requirement such as perfection guarantees that the set of solutions contains such an outcome. For a simpler example of this, see footnote 3 in De Sinopoli (1999).

5 Example 3

Let us consider the AV game (Γ) for the following example:

Example 3 There are three voters and four candidates. Utilities are given by

$$u^1 = (1000, 867, 866, 0), \quad u^2 = (115, 1000, 0, 35), \quad u^3 = (0, 35, 115, 1000).$$

This game has a stable set in which player 1 approves the first and the third candidate. Hence strategic stability does not imply sincerity. Moreover, this result still holds in a neighborhood of the game (an open set of payoffs around the considered ones) and also for stronger solution concepts.

Proposition 5 *The strategy combination*

$$s = \left((1,0,1,0), \frac{1}{4}(0,1,0,0) + \frac{3}{4}(1,1,0,0), \frac{1}{4}(0,0,0,1) + \frac{3}{4}(0,0,1,1)\right)$$

forms a stable set of Γ. *Moreover, there exists a neighborhood* (Ψ_Γ) *of* Γ, *in the space of approval games with three voters and four candidates, such that every game in* Ψ_Γ *has a stable set with the same support as* s.

The proof of the Proposition, which is postponed to the Appendix, consists in showing that the equilibrium s is strongly stable (Kojima et al. 1985) and, hence, forms a stable set. The strong stability of s is proven by showing that s is quasi-strict and isolated and, furthermore, that (s_2, s_3) is strongly stable in the 2×2 game obtained by eliminating all the strategies that are not best replies. This proof actually implies the stronger result that s is a regular equilibrium (Harsanyi 1973), because the characterization theorem of Kojima, et al. shows that an equilibrium is regular if and only if it is quasi strict and strongly stable.[6]

Unlike Examples 1 and 2, this game cannot be solved by iterative elimination of dominated strategies. Because $\{s\}$ is a stable set the pure strategies which are played with positive probability in s cannot be eliminated. Hence, here, elimination of dominated strategies cannot eliminate the pure strategies played by players 2 and 3.

Notice in this example, the second candidates is the Condorcet winner and is elected with probability $\frac{1}{64}$ in the equilibrium s, while the third candidate, who is the Condorcet loser, is elected with probability $\frac{31}{64}$.[7] Under plurality rule,

[6] Dutta and Laslier (2005) give a direct but longer proof that s is regular. Even if we obtain the stronger result that s is strongly stable (and regular), we prefer to state the results in terms of strategic stability because many games, including AV games, have no strongly stable equilibria. Consider the example where everybody has the same preference order over the alternatives. In this case, with three or more voters, no strongly stable equilibrium exists.

[7] The probabilities of election of the first and the fourth candidate are, respectively, $\frac{31}{64}$ and $\frac{1}{64}$.

for the same preferences, there is only a stable set[8] and, in this stable set, the Condorcet winner is elected with probability 1.

Furthermore notice that, by adding two voters i and j, with $u^i = (1, 1, 0, 0)$ and $u^j = (0, 0, 1, 1)$, we obtain a strongly stable equilibrium in which the strategies of the original players are the ones in s while i and j use their dominant strategies. Replicating this, we can obtain an example of insincere voting with any odd number of voters. [9]

Our proof also shows that not even more demanding criteria such as strong stability or regularity, can exclude insincere strategies. This is due to the fact that an insincere strategy can be the only best reply to mixed strategy combinations of the opponents. Hence, as long as we allow for mixed strategies, there is no reason to exclude non-sincere behavior.

6 Conclusion

In this paper, three examples of approval voting games have been proposed. The first one allows us to conclude that in the class of approval games, the perfect equilibrium concept is not restrictive enough to capture sophisticated voting, since there are "perfect equilibrium" outcomes that do not survive the iterated elimination of dominated strategies and that are not induced by any stable set. Furthermore, even if there is a Condorcet winner, strategic stability, as well as sophisticated voting, does not imply his election and, as a second example shows, it is possible that nobody votes for him.

The third example shows that strategic stability does not imply sincerity. It is not difficult to see that for every pure strategy of the other players, the set of best replies contains a sincere strategy. As soon as we allow for mixed strategies, not only is this not true, but even a strong requirement such as strategic stability cannot exclude the use of non-sincere strategies. Moreover, this result holds in a complete neighborhood of the game and also for more demanding criteria.

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Appendix

Proof of Proposition 2 Given that a strongly stable equilibrium (Kojima et al. 1985) is a stable set as a singleton,[10] it is enough to prove that

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8 In the unique stable set, players 1 and 2 vote for the second candidate and player 3 for the fourth.

9 Similar examples can be constructed also with 4 voters (see the second example in De Sinopoli, 1999), and, hence, for any number of voters greater than or equal to 3.

10 See Mertens (1991,pp.697–699) which shows how the continuity of the map from the space of perturbed games to subsets of equilibria is a stronger requirement than the one included in the definition of stability.

$$s = \left((1,0,1,0), \frac{1}{4}(0,1,0,0) + \frac{3}{4}(1,1,0,0), \frac{1}{4}(0,0,0,1) + \frac{3}{4}(0,0,1,1) \right)$$

is strongly stable.

The **first step** of the proof consists in showing that s is a quasi-strict equilibrium (each player uses all his pure best replies). To this end we calculate the probability, under s, of each contingency a player can face and, from these probabilities, the expected utility derived from each undominated strategy. It is easy to see that no dominated strategy is a best reply to s.

Player 1
$\Pr(\omega_{-1} = (1,1,1,1) \mid s_{-1}) = \frac{9}{16}$
$\Pr(\omega_{-1} = (0,1,1,1) \mid s_{-1}) = \frac{3}{16}$
$\Pr(\omega_{-1} = (1,1,0,1) \mid s_{-1}) = \frac{3}{16}$
$\Pr(\omega_{-1} = (0,1,0,1) \mid s_{-1}) = \frac{1}{16}$.
From these probabilities it follows that:
$U^1((1,0,1,0), s_{-1}) = \frac{9}{16} \cdot \frac{1866}{2} + \frac{3}{16} \cdot 866 + \frac{3}{16} \cdot 1000 + \frac{1}{16} \cdot \frac{2733}{4} = \frac{58713}{64}$
$U^1((1,0,0,0), s_{-1}) = \frac{9}{16} \cdot 1000 + \frac{3}{16} \cdot \frac{2733}{4} + \frac{3}{16} \cdot 1000 + \frac{1}{16} \cdot \frac{1867}{3} = \frac{176065}{192}$
$U^1((1,1,0,0), s_{-1}) = \frac{9}{16} \cdot \frac{1867}{2} + \frac{3}{16} \cdot 867 + \frac{3}{16} \cdot \frac{1867}{2} + \frac{1}{16} \cdot 867 = \frac{7335}{8}$
$U^1((1,1,1,0), s_{-1}) = \frac{9}{16} \cdot \frac{2733}{3} + \frac{3}{16} \cdot \frac{1733}{2} + \frac{3}{16} \cdot \frac{1867}{2} + \frac{1}{16} \cdot 867 = \frac{7233}{8}$.
Since no dominated strategy is a best reply to s_{-1} we have that $(1,0,1,0)$ is the only best reply to s_{-1} (although this strategy is not sincere).
Player 2
$\Pr(\omega_{-2} = (1,0,1,1) \mid s_{-2}) = \frac{1}{4}$
$\Pr(\omega_{-2} = (1,0,2,1) \mid s_{-2}) = \frac{3}{4}$.
From these probabilities it follows that:
$U^2((0,1,0,0), s_{-2}) = \frac{1}{4} \cdot \frac{1150}{4} + \frac{3}{4} \cdot 0 = \frac{575}{8}$
$U^2((1,1,0,0), s_{-2}) = \frac{1}{4} \cdot 115 + \frac{3}{4} \cdot \frac{115}{2} = \frac{575}{8}$
$U^2((0,1,0,1), s_{-2}) = \frac{1}{4} \cdot 35 + \frac{3}{4} \cdot \frac{35}{2} = \frac{175}{8}$
$U^2((1,1,0,1), s_{-2}) = \frac{1}{4} \cdot \frac{150}{2} + \frac{3}{4} \cdot \frac{150}{3} = \frac{225}{4}$.
Hence, $(0,1,0,0)$ and $(1,1,0,0)$ are the only two pure best replies to s_{-2}.
Player 3
$\Pr(\omega_{-3} = (1,1,1,0) \mid s_{-3}) = \frac{1}{4}$
$\Pr(\omega_{-3} = (2,1,1,0) \mid s_{-3}) = \frac{3}{4}$.
From these probabilities it follows that:
$U^3((0,0,0,1), s_{-3}) = \frac{1}{4} \cdot \frac{1150}{4} + \frac{3}{4} \cdot 0 = \frac{575}{8}$
$U^3((0,0,1,1), s_{-3}) = \frac{1}{4} \cdot 115 + \frac{3}{4} \cdot \frac{115}{2} = \frac{575}{8}$
$U^3((0,1,0,1), s_{-3}) = \frac{1}{4} \cdot 35 + \frac{3}{4} \cdot \frac{35}{2} = \frac{175}{8}$
$U^3((0,1,1,1), s_{-3}) = \frac{1}{4} \cdot \frac{150}{2} + \frac{3}{4} \cdot \frac{150}{3} = \frac{225}{4}$.
Hence, the only two pure best replies of player 3 are $(0,0,0,1)$ and $(0,0,1,1)$.

The **second step** requires to prove that the quasi-strict equilibrium s is isolated. To analyze the set of equilibria near s we can limit the analysis to the case in which the strategy of player 1 is fixed, because he is using a strict best reply.

Moreover, because s is quasi strict, also players 2 and 3 can use (sufficiently close to s) only the pure strategies in s. Hence, to show that s is isolated it is enough to study the equilibria of the following game between players 2 and 3:

	$(0,0,0,1)$	$(0,0,1,1)$
$(0,1,0,0)$	$\frac{575}{2},\frac{575}{2}$	$0,115$
$(1,1,0,0)$	$115,0$	$\frac{115}{2},\frac{115}{2}$

This game has two pure equilibria, i.e. $((0,1,0,0),(0,0,0,1))$ and $((1,1,0,0),(0,0,1,1))$, and a completely mixed one corresponding to s. Hence, s is isolated.

The **third step** consists in showing that s is a strongly stable equilibrium. Since s is quasi-strict and isolated we can conclude (cf. van Damme, 1991: 55, Th. 3.4.4) that (s_2,s_3) is a strongly stable equilibrium of the reduced game where we take s_1 as being fixed. Since the first player is using his strict best reply, s is a strongly stable equilibrium of the whole game. Hence, $\{s\}$ is a stable set of Γ.

The second part of the proposition directly follows from corollary 4.1 in Kojima et al. (1985), which states that, given a game and a strongly stable equilibrium, the unique nearby equilibrium of a nearby game is strongly stable too. In this statement, a game is a point in the Euclidean space of dimension $n\prod_{i=1}^{n} k_i$, where k_i is the number of pure strategies for player i (here $k_i = 2^K$ for all i). The space of approval voting games is a subspace (of dimension nK) defined by the utility of each of the K candidates for each of the n players. Since each "approval game" near Γ has a normal form close to that of Γ and since for sufficiently close games and sufficiently close strategies, no other strategy than the ones in s can be a best reply, the claim easily follows. □

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